

A NOTE ON LAGRANGIAN VANISHING SPHERE BUNDLES

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1. INTRODUCTION

The study of Lagrangian submanifolds (and the restrictions imposed on them) plays a pivotal role in Symplectic topology. For this reason it is good to have an abundance of examples of Lagrangian submanifolds. Generating such examples, however, is not always a simple task as, naturally, each such example involves a unique construction.

A classical way to construct a Lagrangian in a symplectic manifold Σ , which in a sense is considered as part of the "folklore" of the field, is to let Σ appear as a smooth fiber in a Lefschetz fibration. If this is possible the singularities of the fibration induce Lagrangian spheres in Σ and these spheres, in turn, are representatives of the corresponding vanishing cycles in the homology of Σ , see for instance [1, 12].

In this short work our aim is twofold: The first would be describing a generalization of the above mentioned construction to the "Morse-Bott" case. This would lead, whenever such a degeneration exists, to the existence of Lagrangian sphere bundles rather than just spheres. The second part of the paper would be to study the topological restrictions on such Lagrangian sphere bundles implied by the theory of Floer homology for Lagrangian intersections and to illustrate the techniques involved.

Let X be a Kähler manifold and let Ω be a Kähler form on X which we will assume from now on to be fixed. Our setting in this paper is the following:

Definition: A proper holomorphic map $\pi : X \rightarrow \mathbb{D}$ map from a Kähler manifold X to the unit disc $\mathbb{D} \subset \mathbb{C}$ is called a Morse-Bott degeneration of a Kähler manifold Σ if:

- (1) The only critical value of π is $0 \in \mathbb{D}$ and $\text{Crit}(\pi)$ is a submanifold of X .
- (2) The holomorphic Hessian of π is a non-degenerate quadratic form when restricted to the normal bundle of $\text{Crit}(\pi)$ in X .
- (3) Σ is biholomorphic to a nonsingular fiber of π .

Note that if $\text{Crit}(\pi)$ is a point the above definition reduces to give a description of the local model of a classical Lefschetz fibration around a singularity. Our first result is hence the following:

Theorem A: Let $\pi : X \rightarrow \mathbb{D}$ be a Morse-Bott degeneration and let $L \subset \text{Crit}(\pi)$ be a compact Lagrangian submanifold. Every smooth fiber Σ of π contains a Lagrangian submanifold $N(L)$ diffeomorphic to a sphere bundle over L .

We shall refer to the Lagrangian submanifold $N(L)$ as the Lagrangian "vanishing necklace" over L induced by the degeneration π . First, we have the following straightforward example of a Morse-Bott degeneration:

Example: Let $\pi : X \rightarrow \mathbb{D}$ be a Morse-Bott degeneration of Σ with an isolated singularity (i.e a Lefschetz fibration) and let $p : E \rightarrow X$ be a holomorphic projective bundle on X of rank k . Consider the map $\tilde{\pi}_E : E \rightarrow \mathbb{D}$ given by $\tilde{\pi}_E(x) = \pi(p(x))$. Then $\tilde{\pi}_E$ is a Morse-Bott degeneration with $\text{Crit}(\tilde{\pi}_E) = \mathbb{P}^k$.

From now on let us consider the case of a Morse-Bott degeneration with $\text{Crit}(\pi) = \mathbb{P}^k$. Note that the fiber of the Morse-Bott degeneration with $\text{Crit}(\pi) = \mathbb{P}^k$ described above is by itself a projective bundle. It seems, however, that if we require $h^{1,1}(\Sigma) = 1$ an example of such a degeneration is increasingly hard to find. Indeed, in the second part of the paper we will apply methods of symplectic topology, particularly the theory of Floer homology of Lagrangian submanifolds, to show that Morse-Bott degenerations with $\text{Crit}(\pi) = \mathbb{P}^k$ admit non-trivial restrictions.

We shall first apply Theorem A to show that under the assumption $\text{Crit}(\pi) = \mathbb{P}^k$ one can find a Kähler manifold (A, ω) such that $\dim_{\mathbb{C}} A = k + 1$ and $\Sigma \times A$ admits a displaceable Lagrangian necklace $N(L)$ which is topologically a sphere bundle over S^{2k+1} . The construction of this Lagrangian necklace is described in detail in section 4. One main feature of the Lagrangian $N(L)$ is that due to the affine factor it would be hamiltoninally displaceable from itself. In particular, by the fundamental properties of Floer homology, whenever $N(L)$ admits a well defined Floer homology algebra $HF(N(L), N(L))$, it would inevitably be trivial.

Recall that a Kähler manifold is said to be Fano if $c_1(\Sigma)$, the first Chern class of Σ , can be represented by a Kähler form. One way to ensure that the Floer homology algebra $HF(N(L), N(L))$ is well defined is to require the of monotonicity of $N(L)$, see [10, 11] which would hold if we require Σ to be a Fano manifold with $h^{1,1}(\Sigma) = 1$. Assume that

this is indeed the case and define the number

$$C_\Sigma := \min \{ \langle c_1(\Sigma), A \rangle \mid A \in \pi_2(\Sigma) \text{ such that } 0 < \langle c_1(\Sigma), A \rangle \}$$

to which we refer as the minimal Chern number of Σ . We obtain the following restrictions:

Theorem B: Let Σ be a Fano manifold of $\dim_{\mathbb{C}} \Sigma = n$ with $h^{1,1}(\Sigma) = 1$ and minimal Chern number C_Σ . Suppose that Σ appears as a fibre of a Morse-Bott degeneration $\pi : X \rightarrow \mathbb{D}$ with $\text{Crit}(\pi) \simeq \mathbb{P}^k$ and $k \neq 0, n-1$. The following restrictions hold:

(1) If $n \neq 3k+1$ then one of the following holds:

(a) $C_\Sigma \mid k+1$.

(b) $2C_\Sigma \mid n-k+1$.

(c) $2C_\Sigma \mid n+k+2$. Furthermore $C_\Sigma \mid 2k+1$ if $n > 3k+1$ and $C_\Sigma \mid 2k+2$ if $n < 3k+1$.

(2) If $n = 3k+1$ then $C_\Sigma \mid n-k+1$.

It is interesting to note that we have the following example of a Morse-Bott degeneration with $\text{Crit}(\pi) = \mathbb{P}^{n-1}$, which is the case not covered by our theorem:

Example: Let Σ be a Kähler manifold of dimension $\dim_{\mathbb{C}} \Sigma = n$ and let $Y \subset \Sigma$ be a submanifold. Consider the manifold $X = \Sigma \times \mathbb{D}$ and let $p_Y : \tilde{X}_Y \rightarrow X$ be the blow up of X along the $Y \times \{0\}$. Denote by E be the exceptional divisor of \tilde{X}_Y . The map $\pi : \tilde{X}_Y \rightarrow \mathbb{D}$ given by $\pi = pr_{\mathbb{D}} \circ p_Y$ is a Morse-Bott degeneration of Σ with $\text{Crit}(\pi) = E$. In particular, if Y is a point $\text{Crit}(\pi) \simeq \mathbb{P}^{n-1}$.

In view of Theorem B, it is tempting to suggest that it is impossible to include a Fano manifold Σ with $h^{1,1}(\Sigma) = 1$ as a fibre of a Morse degeneration with $\text{Crit}(\pi) = \mathbb{P}^k$. Finally, let us conclude with the following remark:

Remark (Degenerations and pencils): In projective geometry, degenerations with isolated singularities exist in abundance as they arise naturally from Lefschetz pencils which could be associated to any projective manifold X . It is interesting to note that such a straightforward correspondence between degenerations and pencils does not seem to hold in the non-isolated, Morse-Bott case, due to what appears to be essential reasons. This would be discussed in detail at the end of the paper.

The rest of the paper is organized as follows: In section 2 we prove Theorem A. Section 3 is devoted to an overview of required results on Floer homology of monotone Lagrangian submanifolds. In section 4 we discuss Morse-Bott degenerations with $\text{Crit}(\pi) = \mathbb{P}^k$ and prove Theorem B. We discuss the obstructions to obtain Morse-Bott degenerations from pencils in section 5.

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2. PROOF OF THE LAGRANGIAN NECKLACE THEOREM

This section is devoted to the proof of the following theorem:

Theorem 2.1: Let $\pi : X \rightarrow \mathbb{D}$ be a Morse-Bott degeneration and let $L \subset \text{Crit}(\pi)$ be a Lagrangian submanifold. Every smooth fiber $\Sigma_z = \pi^{-1}(z)$ for $0 \neq z \in \mathbb{D}$ contains a Lagrangian submanifold $N_z(L)$ diffeomorphic to a sphere bundle over L .

We follow the lines of Donaldson's proof for the Morse case which appears in [12]. We refer the reader to [3] for relevant facts on Morse-Bott theory. Set $F = \text{Re}(\pi)$ and $H = \text{Im}(\pi)$ and denote by J the complex structure of X . Since π is holomorphic the negative gradient flow of F with respect to the metric $g_J(\cdot, \cdot) = \Omega(\cdot, J\cdot)$ is the same as the Hamiltonian flow of H with respect to the Kähler form Ω . Denote this flow by ϕ_t and let $S(L)$ be the stable submanifold of L under this flow, see [9].

Lemma 2.2: $S(L)$ is a lagrangian submanifold in (X, Ω) .

Proof: Let $\Theta : S(L) \rightarrow L$ be the end point map given by $\Theta(x) := \lim_{t \rightarrow \infty} \phi_t(x)$. In a small neighborhood $U \subset S(L)$ of L the map Θ is a locally trivial fibration over L , see [9]. Let us show that U is isotropic. Indeed, for every $x \in U$ and $v_1, v_2 \in T_x(U)$ we have

$$\Omega_x(v_1, v_2) = \Omega_{\pi(x)}(d\Theta(v_1), d\Theta(v_2)) = 0$$

because ϕ_t is Hamiltonian with respect to Ω and L is Lagrangian. Moreover, the fact that U is isotropic implies that $S(L)$ is isotropic since ϕ_t is Hamiltonian and every point $x \in S(L)$ is sent to U by ϕ_t for t large enough.

Finally recall that

$$\dim_{\mathbb{R}} S(L) = \dim_{\mathbb{R}} L + \text{index}(F)$$

Where $\text{index}(F)$ is the Morse-Bott index, see [3]. A simple computation shows that

$$\text{index} F = \frac{1}{2} (\dim_{\mathbb{R}} X - \dim_{\mathbb{R}} \text{Crit}(\pi)) = \frac{1}{2} \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} L$$

Thus $\dim_{\mathbb{R}} S(L) = \frac{1}{2} \dim_{\mathbb{R}} X$ and hence $S(L)$ is Lagrangian. \square

Continuation of the proof of Theorem 2.1: Denote $N_{\epsilon}(L) = F^{-1}(\epsilon) \cap S(L)$ for $\epsilon > 0$ small enough. By Morse-Bott theory $N_{\epsilon}(L)$ is a sphere bundle over L , of dimension $\dim_{\mathbb{R}} N_{\epsilon}(L) = \dim_{\mathbb{R}} S(L) - 1$. As ϕ_t is the Hamiltonian flow of H , the function H is constant along the flow lines of ϕ_t . We deduce that $S(L) \subset H^{-1}(0)$. Consequently

$$N_{\epsilon}(L) \subset F^{-1}(\epsilon) \cap H^{-1}(0) = \pi^{-1}(\epsilon) = \Sigma_{\epsilon}$$

This proves that the fibre Σ_{ϵ} contains the Lagrangian submanifold $N_{\epsilon}(L)$ which is a sphere bundle over L . Finally, by Moser argument all fibers $\Sigma_z = \pi^{-1}(z)$, $z \neq 0$, are symplectomorphic, hence they all contain such Lagrangians. \square

Definition 2.2: Let $\pi : X \rightarrow \mathbb{D}$ be a Morse-Bott degeneration and $L \subset \text{Crit}(\pi)$ be a Lagrangian submanifold. We refer to $N_z(L) \subset \Sigma_z$ as the Lagrangian necklace over L in the fibre Σ_z .

Remark: In case $\text{Crit}(\pi)$ is a finite number of points and L coincides with this set of points then $N_z(L)$ is just the set of Lagrangian spheres representing the vanishing cycles as appears in [1, 12].

3. BRIEF OVERVIEW OF FLOER AND QUANTUM HOMOLOGY FOR MONOTONE LAGRANGIAN SUBMANIFOLDS

An essential ingredient in our approach is the theory of Floer homology for Lagrangian submanifolds. This theory was originally introduced by Floer, for some class of Lagrangian submanifolds, in order to study problems of Lagrangian intersections, see e.g [7]. The theory was later extended by Oh [10] for the class of so called monotone Lagrangian submanifolds. We shall not go here into the details of this homology theory, but only indicate our algebraic conventions as well as some essential properties of Floer homology. A detailed account on the subject can be found in [7, 10], see also [4, 5, 6].

Let (M, ω) be a closed symplectic manifold and $L \subset M$ a monotone closed Lagrangian submanifold. Monotone means that for all $\alpha \in \pi_2(M, L)$ we have

$$\omega(\alpha) > 0 \Leftrightarrow \mu(\alpha) > 0$$

Where $\mu : \pi_2(M, L) \rightarrow \mathbb{Z}$ is the Maslov class homomorphism. We denote by

$$N_L = \min \{ \mu(\alpha) \mid \mu(\alpha) > 0, \alpha \in \pi_2(M, L) \}$$

The minimal Maslov index of L , so that $\text{image}(\mu) \subset N_L \cdot \mathbb{Z}$. We shall assume from now on that L is monotone and $N_L \geq 2$. Under these assumptions one can associate to L

an important invariant called the Floer homology $HF_*(L, L)$. This is a graded vector space over the field \mathbb{Z}_2 . It is the homology of a certain chain complex defined by counting solutions to a (non-linear) elliptic PDE with boundary conditions on L (called pseudo-holomorphic curves). The resulting homology measures the obstruction to Hamiltonianly displace L , i.e for the existence of $\varphi \in Ham(M, \omega)$ s.t $\varphi(L) \cap L = \emptyset$.

Below we shall use essentially Oh's version of Floer homology as developed in [10] only that our $HF_*(L, L)$ will be \mathbb{Z} -graded as in [4, 5, 6] and N_L -periodic, i.e $HF_{*+N_L}(L, L) = HF_*(L, L)$.

Denote by $\Lambda = \mathbb{Z}_2[t^{-1}, t]$ the ring of Laurent polynomials in t with coefficients in \mathbb{Z}_2 . We grade Λ by declaring the degree of t to be $|t| = -N_L$. Then our $HF_*(L, L)$ becomes a graded module over Λ , and we have

$$t \cdot HF_i(L, L) = HF_{i-N_L}(L, L)$$

For all $i \in \mathbb{Z}$.

Call a Lagrangian submanifold $L \subset M$ *displaceable* if there exists a Hamiltonian diffeomorphism $\varphi \in Ham(M, \omega)$ such that $\varphi(L) \cap L = \emptyset$. In what follows we shall use the following important vanishing property of HF : *If L is displaceable then $HF(L, L) = 0$.*

Another important ingredient in our Floer-theoretical considerations is a spectral sequence which is useful for computing HF . This sequence was introduced by Oh [11], but here we shall follow a variant of it as described by Biran in [4]. The main properties of this sequence that are relevant for our applications are listed in the following theorem.

Theorem (see Oh [11], Biran [4]): There exists a spectral sequence $\{E_{p,q}^r, d^r\}_{r \geq 1}$ with the following properties:

- (1) $E_{p,q}^1 \simeq H_{p+q-N_L}(L; \mathbb{Z}_2) \cdot t^{-p}$ for every p, q .
- (2) $E_{p+1,q}^r \simeq E_{p,q+1-N_L}^r t^{-1}$ for every p, q, r .
- (3) $\{E_{p,q}^r, d^r\}$ collapses after a finite number of pages and converges to $HF_*(L, L)$.

In particular, if L is displaceable then $E_{p,q}^\infty = 0$ for every p, q .

4. MORSE-BOTT DEGENERATIONS WITH CRITICAL LOCUS \mathbb{P}^k

Let Σ be a Kähler manifold of $\dim_{\mathbb{C}} \Sigma = n$. In this section we consider the special case of Morse-Bott degenerations $\pi : X \rightarrow \mathbb{D}$ of Σ with $Crit(\pi) \simeq \mathbb{P}^k$. Let $B^{2n}(r) \subset \mathbb{C}^n$ be the standard symplectic ball of radius $r > 0$ with symplectic form ω_0 induced by the standard symplectic form on \mathbb{C}^n and let $S^{2n-1}(r) = \partial B^{2n}(r)$ be the corresponding sphere. In [2]

it was observed that, for any $r > 1$, the manifold $\mathbb{P}^k \times B^{2k+2}(r)$ contains the embedded Lagrangian sphere

$$L = \{(h(z), \bar{z}) \mid z \in S^{2k+1}(1)\} \subset (\mathbb{P}^k \times B^{2k+2}(r), \omega_{FS} \oplus \omega_0)$$

where $h : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ is the Hopf map given by $z \mapsto \mathbb{C} \cdot z$ and ω_{FS} is the Fubini-Study form on \mathbb{P}^k normalized such that $\int_{\mathbb{P}^1} \omega_{FS} = \pi$.

Let (A, ω) be a compact closed symplectic manifold of $\dim_{\mathbb{C}} A = k + 1$ such that there exists a symplectic embedding of the standard symplectic ball $(B^{2k+2}(r), \omega_0)$ into A for $r \gg 0$. Consider the stabilization

$$\tilde{\pi} : X \times A \rightarrow \mathbb{D}$$

of the map of π given by $(z, u) \mapsto \pi(z)$. This induces a Morse-Bott degeneration of $\Sigma \times A$ with $\text{Crit}(\tilde{\pi}) = \mathbb{P}^k \times A$. Thus, for any Kähler form Ω on X such that $\Omega|_{\mathbb{P}^k} = \omega_{FS}$ we get a Lagrangian sphere $L \subset \text{Crit}(\tilde{\pi})$ and by Theorem 2.1 we have a Lagrangian Necklace $N(L) \subset (\Sigma \times A, \Omega|_{\Sigma} \times \omega)$. We have:

Proposition 4.1: The Lagrangian necklace $N(L) \subset \Sigma \times A$ is a Hamiltonianly displaceable, simply connected, Lagrangian sub-manifold which is topologically a sphere bundle over S^{2k+1} .

Proof: Note that the Lagrangian sphere $L \subset \mathbb{P}^k \times B^{2k+2}(4)$ can be displaced from itself by Hamiltonian isotopy φ which is the identity on the \mathbb{P}^k factor and acting as a Hamiltonian isotopy in the $B^{2k+2}(4)$ factor which displaces $B^{2k+2}(1) \subset B^{2k+2}(4)$ from itself. Denote by $L' = \varphi(L)$. In particular, for $\epsilon > 0$ small enough, $N_{\epsilon}(L) \cap N_{\epsilon}(L') = \emptyset$ since $L \cap L' = \emptyset$. \square

We further have:

Lemma 4.3: Let Σ be a Fano manifold with $\dim_{\mathbb{C}} \Sigma = n$ which satisfies $h^{1,1}(\Sigma) = 1$. Assume Σ appears as a fibre in a degeneration $\pi : X \rightarrow \mathbb{D}$ with $\text{Crit}(\pi) = \mathbb{P}^k$. $1 \leq k \leq n - 2$ then

$$HF(N(L), N(L)) = 0$$

Proof: The condition $h^{1,1}(\Sigma) = 1$ assures that the form on Σ is monotone. In particular, since under the above conditions $N(L)$ is simply connected, it would be a monotone Lagrangian submanifold of Σ . Hence $N(L)$ has a well defined Floer homology. By Proposition 4.1 $N(L)$ is Hamiltonianly displaceable and thus $HF(N(L), N(L)) = 0$. \square

We are now in position to prove the following theorem:

Theorem 4.4: Let Σ be a Fano manifold of $\dim_{\mathbb{C}} \Sigma = n$ with $h^{1,1}(\Sigma) = 1$ and minimal Chern number C_{Σ} . Suppose that Σ appears as a fibre of a Morse-Bott degeneration $\pi : X \rightarrow \mathbb{D}$ with $\text{Crit}(\pi) \simeq \mathbb{P}^k$ and $k \neq 0, n-1$. The following restrictions hold:

(1) If $n \neq 3k+1$ then one of the following holds:

(a) $C_{\Sigma} \mid k+1$.

(b) $2C_{\Sigma} \mid n-k+1$.

(c) $2C_{\Sigma} \mid n+k+2$. Furthermore $C_{\Sigma} \mid 2k+1$ if $n > 3k+1$ and $C_{\Sigma} \mid 2k+2$ if $n < 3k+1$.

(2) If $n = 3k+1$ then $C_{\Sigma} \mid n-k+1$.

Proof: First, by standard considerations, the Maslov index of $N(L)$ satisfies $N_L = 2C_{\Sigma}$. By the Gysin sequence for sphere bundles, the singular homology $H_i(N(L), \mathbb{Z}_2) = 0$ can only be nonzero if

$$i = 0, 2k+1, n-k, n+k+1$$

Let us consider the spectral sequence described in section 3. Taking $p = q = 0$, the differential of the spectral sequence has the following form

$$\dots \rightarrow E_{r,-r+1}^r \rightarrow E_{0,0}^r \rightarrow E_{-r,r-1}^r \rightarrow \dots$$

For $r \geq 0$ Moreover, for degree reasons we have $E_{r,-r+1}^r = 0$ for all $r \geq 0$. Thus, since $E_{0,0}^1 \simeq \mathbb{Z}_2$, in order for $E_{0,0}^{r+1}$ to vanish for some $r \geq 1$ we must have $E_{-r,r-1}^r \neq 0$. But this can only happen in one of the following cases:

(i) $2rC_{\Sigma} - 1 = 2k+1$ for some $r \geq 0$.

(ii) $2rC_{\Sigma} - 1 = n-k$ for some $r \geq 0$.

(iii)

(iii.1) $2r_1C_{\Sigma} - 1 = n+k+1$ for some $r \geq 0$.

(iii.2.1) $(2k+1) + 2r_2C_{\Sigma} - 1 = n-k$ if $3k+1 < n$.

(iii.2.2) $(n-k) + 2r_2C_{\Sigma} - 1 = 2k+1$ if $n < 3k+1$.

Finally, note that (iii) can hold only if $n \neq 3k+1$ and in the complementary case (i) and (ii) coincide. This proves (1) and (2). \square

5. ON THE CONSTRUCTION OF MORSE-BOTT DEGENERATIONS

A common way to obtain degenerations with isolated singular points is by considering Lefschetz fibrations which in turn arise from Lefschetz pencils. In the non-isolated case, however, there seems to be no such straightforward transition from pencils to fibrations. In this section we would like to explain this difference between the isolated and non-isolated cases. We refer the reader to [14] for a complete treatment on the geometry of Lefschetz pencils.

Let $X \subset \mathbb{P}^N$ be an algebraic manifold and denote by $(\mathbb{P}^N)^*$ the dual projective space parametrizing hyperplanes in \mathbb{P}^N . To X one can associate the discriminant variety given by

$$X^* = \{H \mid \Sigma_H \text{ is singular}\} \subset (\mathbb{P}^N)^*$$

where $\Sigma_H = X \cap H$ is the hyperplane section corresponding to H . A pencil on X is a line $\ell \subset (\mathbb{P}^N)^*$. Let ℓ be a pencil on X and define the variety

$$\tilde{X}_\ell = \{(x, H) \mid x \in \Sigma_H\} \subset X \times \ell$$

and denote by $\pi : \tilde{X}_\ell \rightarrow \ell \simeq \mathbb{P}^1$ the map given by projection on the ℓ factor. Let

$$B(\ell) = \bigcap_{H \in \ell} \Sigma_H \quad ; \quad S(\ell) = \bigcup_{H \in \ell} \text{Sing}(\Sigma_H)$$

be the base locus and singular locus of ℓ respectively. We have:

Proposition 5.1: $\text{Sing}(\tilde{X}_\ell) = (B(\ell) \cap S(\ell)) \times (\ell \cap X^*)$.

Proof: Let (z, t) be local coordinates in a neighborhood $U \times \mathbb{C} \subset X \times \ell$ around (x, H_0) such that:

$$\tilde{X}_\ell \cap (U \times \mathbb{C}) = \{(z, t) \mid f_0(z) + tf_1(z) = 0\}$$

where

$$\Sigma_{H_0} \cap U = \{f_0(z) = 0\} \quad , \quad \Sigma_{H_1} \cap U = \{f_1(z) = 0\}$$

with $H_1 \in \ell$ a hyperplane different from H_0 . The origin is thus a singular point if and only if $f_0(0) = f_1(0) = 0$ and $df_0(0) = 0$. \square

In particular, since $B(\ell) \subset \Sigma_H$ is an ample divisor, we have:

Corollary 5.2: \tilde{X}_ℓ is singular if $\dim(S(\ell)) \geq 1$.

Note that by definition $S(\ell) = \bigcup_{H \in \ell \cap X^*} \text{Sing}(\Sigma_H)$. Recall that a pencil ℓ is said to be a Lefschetz pencil on X if it intersects the discriminant variety X^* transversally. Of course, for such an intersection to exist and be non-void we have to assume that $\text{codim}(X^*) = 1$. It is well known that if ℓ is a Lefschetz pencil the map $\pi : \tilde{X}_\ell \rightarrow \ell$, referred to as a Lefschetz

pencil, has only isolated singular points and satisfies the Morse(-Bott) conditions locally around these point. In contrast, when $\text{codim}(X^*) \geq 2$ a line $\ell \subset (\mathbb{P}^N)^*$ will either be disjoint from X^* , or intersect X^* non-transversally. In the latter case we obtain $\dim(S(\ell)) \geq 1$.

Thus, by Corollary 5.2 if $\dim(S(\ell)) \geq 1$ the map $\pi : \tilde{X}_\ell \rightarrow \ell$ itself cannot give rise to a Morse-Bott degeneration simply because \tilde{X}_ℓ will be singular. One can, however, attempt to resolve the singularities of \tilde{X}_ℓ .

Example: Let $z = [z_0 : \dots : z_n]$ be homogenous coordinates on $X = \mathbb{P}^n$ and consider the pencil ℓ of quadrics given by

$$\Sigma_\lambda = \{\lambda_0 F_0(z) + \lambda_1 F_1(z) = 0\} \subset \mathbb{P}^n$$

Where

$$F_0(z) = z_0^2 + \dots + z_n^2 \quad ; \quad F_1(z) = -(a_2 z_2^2 + a_3 z_3^2 + \dots + a_n z_n^2)$$

Where the coefficients a_i are non-zero and $a_i \neq a_j$ for all $2 \leq i, j \leq n$. In particular, the quadric $\Sigma_{[0:1]}$ is singular with

$$\text{Sing}(\Sigma_{[0:1]}) = \{z_2 = \dots = z_n = 0\}$$

Moreover, since the rest of the singular fibers are isolated and disjoint of the base locus we have

$$B(\ell) \cap S(\ell) = B(\ell) \cap \text{Sing}(\Sigma_{[0:1]}) = \left\{ \begin{array}{l} z_0^2 + z_1^2 = 0 \\ z_2 = \dots = z_n = 0 \end{array} \right\}$$

which constitutes of the two points $z_\pm = [1, \pm i, 0, \dots, 0]$. If one would blow up \tilde{X}_ℓ along the two points $\{z_-, z_+\} \times \ell$ one would obtain a degeneration. However, note that the fibres of this degeneration would, at this point, be a quadric blown up at two points and not the original quadrics we began with.

REFERENCES

- [1] V. Arnold. Some remarks on symplectic monodromy of Milnor fibrations. The Floer memorial volume, volume 133 of Progr. Math., 99–103. Birkhäuser Verlag, Basel, 1995.
- [2] M. Audin, F. Lalonde, and L. Polterovich. Symplectic rigidity: Lagrangian submanifolds. In M. Audin and J. Lafontaine, editors, Holomorphic curves in symplectic geometry, volume 117 Progr. Math., pages 271–321, Basel, 1994. Birkhäuser Verlag.
- [3] D. M. Austin and P. J. Braam. Morse-Bott theory and equivariant cohomology. The Floer memorial volume, volume 133 Progr. Math., pages 123–183. Birkhäuser Verlag, Basel, 1995.
- [4] P. Biran. Lagrangian non-intersections *Geom. and Func. Anal.*, 16, 279–326, 2006.
- [5] P. Biran, O. Cornea. Rigidity and Uniruling for Lagrangian Submanifolds. preprint 2008.
- [6] P. Biran, O. Cornea. A Lagrangian Quantum Homology. New perspectives and challenges in symplectic field theory, 1–44, CRM Proc. Lecture Notes, 49, Amer. Math. Soc., Providence, RI, 2009.

- [7] A. Floer. Morse theory for Lagrangian intersections. *J. Differential Geom.*, 28(3):513–547, 1988.
- [8] Y. Jerby. Algebraic families and Lagrangian submanifolds. MSc Thesis, Tel Aviv University, 2004.
- [9] A. Katok , B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press.
- [10] Y.-G. Oh. Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I. *Comm. Pure Appl. Math.*, 46(7):949–993, 1993.
- [11] Y.-G. Oh. Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings. *Internat. Math. Res. Notices*, (7):305–346, 1996.
- [12] P. Seidel. Floer Homology and The Symplectic Isotopy Problem. Phd Thesis 1997
- [13] E. Tevelev. Projectively dual varieties. *J. Math. Sci. (N. Y.)* 117 (2003), no. 6, 4585–4732.
- [14] C. Voisin. *Hodge theory and complex algebraic geometry. II*. Cambridge Studies in Advanced Mathematics, 77. Cambridge University Press, Cambridge, 2003.